

HARDY-LITTLEWOOD TUPLE CONJECTURE OVER LARGE FINITE FIELDS

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ABSTRACT. We prove the following function field analog of the Hardy-Littlewood conjecture (which generalizes the twin prime conjecture) over large finite fields. Let n, r be positive integers and q an odd prime power. For a tuple of distinct polynomials $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{F}_q[t]^r$ of degree $< n$ let $\pi(q, n; \mathbf{a})$ be the number of monic polynomials $f \in \mathbb{F}_q[t]$ of degree n such that $f + a_1, \dots, f + a_r$ are simultaneously irreducible. We prove that $\pi(q, n; \mathbf{a}) \sim \frac{q^n}{n^r}$ as $q \rightarrow \infty$, q odd, and n, r fixed.

1. INTRODUCTION

The twin prime conjecture predicts that there are infinitely many positive integers n such that n and $n + 2$ are primes. In other words, if

$$\pi_2(x) = \#\{1 \leq n \leq x \mid n \text{ and } n + 2 \text{ are primes}\}$$

be the corresponding counting function, then the conjecture says that $\pi_2(x) \rightarrow \infty$ as $x \rightarrow \infty$. A more precise conjecture, the Hardy-Littlewood conjecture, predicts that

$$\pi_2(x) \sim 2 \prod_{p>2} \frac{p(p-2)}{(p-1)^2} \frac{x}{\log^2 x} \approx 1.32 \frac{x}{\log^2 x}, \quad x \rightarrow \infty.$$

More generally, let $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r$ be an r -tuple of distinct integers and let

$$\pi(x; \mathbf{a}) = \#\{1 \leq n \leq x \mid n + a_1, \dots, n + a_r \text{ are all prime}\}$$

is the corresponding prime counting function. Then $\pi(x) = \pi(x; 0)$ and $\pi_2(x) = \pi(x; 0, 2)$. If a_1, \dots, a_r cover all residues modulo some prime p , e.g. $(a_1, a_2, a_3) = (0, 2, 4)$ and $p = 3$, then for any n there is i such that $p \mid n + a_i$. In particular, $\pi(x; \mathbf{a})$ is bounded as $x \rightarrow \infty$. Otherwise the Hardy-Littlewood conjecture [5] says that $\pi(x; \mathbf{a}) \rightarrow \infty$ as $x \rightarrow \infty$.

As in the twin prime conjecture, the Hardy-Littlewood conjecture gives the rate in which $\pi(x; \mathbf{a})$ tends to infinity: Let

$$\nu(p; \mathbf{a}) = \#\{a_1 \pmod p, \dots, a_r \pmod p\}$$

and

$$C(\mathbf{a}) = \prod_p \frac{1 - \nu(p; \mathbf{a})/p}{(1 - 1/p)^r}.$$

It is an exercise in analytic number theory that, unless a_1, \dots, a_r cover all residues modulo some prime p , the product converges, i.e. $C(\mathbf{a}) > 0$. Then the Hardy-Littlewood conjecture predicts that

$$\pi(x; \mathbf{a}) \sim C(\mathbf{a}) \frac{x}{\log^r x}, \quad x \rightarrow \infty.$$

The objective of this paper is to prove an analog of the Hardy-Littlewood conjecture over *large* finite fields of odd cardinality. Let q be a prime power, let \mathbb{F}_q be the finite field of q elements, and let n and r be integers. For an r -tuple of distinct polynomials $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{F}_q[t]^r$, each of degree $< n$, we let

$$\pi(q, n; \mathbf{a}) = \#\{f \in \mathbb{F}_q[t] \mid \begin{array}{l} f \text{ is monic and of degree } n \text{ and} \\ f + a_1, \dots, f + a_r \text{ are all irreducible} \end{array}\}.$$

Since q^n is the number of degree n monic polynomials, it plays the role of x in this setting. Therefore it is desirable to estimate $\pi(q, n; \mathbf{a})$ as $q^n \rightarrow \infty$. The straightforward analog of the classical setting, and the more difficult case, is when q is fixed and $n \rightarrow \infty$. In this paper we treat the asymptotic when n is fixed and $q \rightarrow \infty$:

Theorem 1.1. *Let n and r be positive integers and q an odd prime power. Then for every distinct $a_1, \dots, a_r \in \mathbb{F}_q[t]$, each of degree $< n$, we have*

$$\pi(q, n; \mathbf{a}) = \frac{q^n}{n^r} + O_{n,r}(q^{n-\frac{1}{2}}).$$

Remark. In contrast to the original Hardy-Littlewood conjecture, in Theorem 1.1 the a_1, \dots, a_r do not have to be fixed in advance. Actually even the characteristic of the ring \mathbb{F}_q may vary.

Remark. Our proof may give an explicit bound of the implied constant in the above theorem. However this bound is not so good, as its dependence on n is worse than $n!$. For this reason, in this paper, we avoid tracking this bound.

Remark. In [3] Bender and Pollack prove Theorem 1.1 for $r = 2$.

Remark. Theorem 1.1 extends a special case of a result of Pollack [8] (provided $\gcd(2n, q) = 1$) and the author [1] (provided either q odd or q even and n odd) from constant a_1, \dots, a_r to polynomials.

Outline of the proof. Our approach is generic in the following sense: We take \mathcal{F} to be a degree n monic polynomial with variable coefficients and count in how many ways we can specialize the coefficients of \mathcal{F} to \mathbb{F}_q such that $f + a_1, \dots, f + a_r$ are irreducible, where f is the specialized polynomial.

In order to achieve this we use an irreducibility criterion, whose proof is based on an earlier result of the author reducing the problem to a problem on rational points. Then the proof applies the Lang-Weil estimates (Section 2).

In order to apply the irreducibility criterion we need to calculate the Galois group of $\prod_{i=1}^r (\mathcal{F} + a_i)$. Using a group theoretical lemma, this calculation is reduced to proving square independence of discriminants which is achieved by applying a result of Carmon and Rudnick [4] (Section 3).

2. IRREDUCIBILITY RESULT

We denote by $\tilde{\mathbb{F}}_q$ a fixed algebraic closure of \mathbb{F}_q .

Proposition 2.1. *Let $(\mathbf{U}, t) = (U_1, \dots, U_n, t)$ be an $(n+1)$ -tuple of variables, let $\mathcal{F}_1, \dots, \mathcal{F}_r \in \mathbb{F}_q[\mathbf{U}, t]$ be polynomials each of degree n in t and with respective splitting fields F'_1, \dots, F'_r over $E' = \mathbb{F}_q(\mathbf{U})$, and let $F' = F'_1 \cdots F'_r$. Let $E = E' \cdot \tilde{\mathbb{F}}_q$, and similarly $F_i = F'_i \cdot \tilde{\mathbb{F}}_q$ and $F = F' \cdot \tilde{\mathbb{F}}_q$. Assume that*

$$\text{Gal}(F/E) \cong S_n^r.$$

Then the number of $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{F}_q^n$ for which all the specialized polynomials $f_i(t) = \mathcal{F}_i(\mathbf{u}, t)$ are irreducible is $q^n/n^r + O_{n,r}(q^{n-1/2})$ as $q \rightarrow \infty$ and n, r are fixed.

To prove this we apply [2, Lemma 2.8] (which is a special case of [1, Lemma 2.2]), which we quote here for the reader's convenience:

Lemma 2.2. *Let K be a field, let $(\mathbf{U}, t) = (U_1, \dots, U_n, t)$ an $(n+1)$ -tuple of variables over K . For each $1 \leq i \leq r$ let $\mathcal{F}_i \in K[\mathbf{U}, t]$ be an absolutely irreducible polynomial which is separable and of degree n_i in t . Let L_i be a Galois extension of K of degree n_i .*

We denote the splitting field of $\mathcal{F} = \mathcal{F}_1 \cdots \mathcal{F}_r$, considered as a polynomial in t , over $E' = K(\mathbf{U})$ by F' and assume that F'/K is regular and $\text{Gal}(F'/E') \cong \prod_{i=1}^r S_{n_i}$.

Then there exist a proper algebraic subset Z of \mathbb{A}_K^n , an absolutely irreducible normal affine K -variety W , and a finite étale map $\rho: W \rightarrow V$ with $V = \mathbb{A}_K^n \setminus Z$ such that the following conditions hold:

- (1) *Z and W are defined in \mathbb{A}_K^n and \mathbb{A}_K^{n+3} , respectively, by polynomials with coefficients in K whose degrees are bounded in terms of n, r .*
- (2) *$\deg(\mathcal{F}_i(\mathbf{u}, t)) = n_i$ for each $\mathbf{u} \in V(K)$ and $i = 1, \dots, r$.*
- (3) *$\rho(W(K))$ is the set of all $\mathbf{u} \in V(K)$ such that L_i is generated by a root of $\mathcal{F}_i(\mathbf{u}, t)$, $i = 1, \dots, r$. In particular, $\mathcal{F}_i(\mathbf{u}, t)$ is irreducible of degree n_i for all $\mathbf{u} \in \rho(W(K))$ and $1 \leq i \leq r$.*
- (4) *$|(\rho')^{-1}(\mathbf{u}) \cap W(K)| = \prod_{i=1}^r n_i$ for all $\mathbf{u} \in \rho(W(K))$.*

Remark. In [2, Lemma 2.8] the condition that F' is regular over K was mistakenly omitted. See erratum of [2] for further details.

Proof of Proposition 2.1. Let $K = \mathbb{F}_q$. Since

$$S_n^r \cong \text{Gal}(F/E) \hookrightarrow \text{Gal}(F'/E') \hookrightarrow S_n^r,$$

we get that $\text{Gal}(F/E) \cong \text{Gal}(F'/E') \cong S_n^r$ and thus that F'/K is regular. In particular, each \mathcal{F}_i is absolutely irreducible and separable. For each i , we let $L_i = \mathbb{F}_{q^n}$ be the unique extension of \mathbb{F}_q of degree n .

By Lemma 2.2 (with $n_1 = \dots = n_r = n$) there exist a proper algebraic subset Z of \mathbb{A}_K^n , an absolutely irreducible normal affine K -variety W' , and a finite étale map $\rho': W' \rightarrow V$ with $Z = \mathbb{A}_K^n \setminus V$ that satisfy conditions (1)-(4) of the lemma.

Since Z is a closed subset of \mathbb{A}_K^n defined by polynomials of bounded degrees in terms of n, r we get that $\#Z(K) = O_{n,r}(q^{n-1})$ (see e.g. [7, Lemma 1]). Note that $\dim W' = n$, so the Lang-Weil estimates give

$$\#W'(K) = q^n + O_{n,r}(q^{n-\frac{1}{2}}).$$

(See [7, Theorem 1]. In [9, Theorem 2.1] Zywinia gives the best known bounds on the implied constant.)

Let S be the set of all $\mathbf{u} \in K^n$ such that $\mathcal{F}_i(\mathbf{u}, t)$ is irreducible in $K[t]$, for every $i = 1, \dots, r$. By Lemma 2.2(3), $\rho'(W'(K))$ is the set of all $\mathbf{u} \in V(K) \subseteq K^n$ such that \mathbb{F}_{q^n} is generated by a root of $\mathcal{F}_i(\mathbf{u}, t)$ over K , for every $i = 1, \dots, r$. Since $\mathcal{F}_i(\mathbf{u}, t)$ is of degree n , for $\mathbf{u} \in V(K)$, and \mathbb{F}_{q^n} is the unique extension of degree n of K , we get that $\rho'(W'(K)) = S \cap V(K)$, hence $\#\rho'(W'(K)) = \#(S \cap V(K))$.

Lemma 2.2(4) gives that $\#((\rho')^{-1}(\mathbf{u}) \cap W'(K)) = n^r$, for every $\mathbf{u} \in \rho'(W'(K))$. We therefore get that

$$\begin{aligned} \#S &= \#(S \cap V(K)) + \#(S \cap Z(K)) = \#\rho'(W'(K)) + O_{n,r}(q^{n-1}) \\ &= \frac{1}{n^r} \#W'(K) + O_{n,r}(q^{n-1}) = \frac{q^n}{n^r} + O_{n,r}(q^{n-\frac{1}{2}}), \end{aligned}$$

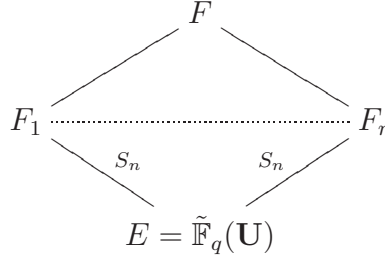
as needed. \square

3. GALOIS GROUPS

Let q be an odd prime power, let n, r be positive integers, let $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{F}_q[t]^r$ be an r -tuple of distinct polynomials each of degree $< n$, let $\mathbf{U} = (U_1, \dots, U_n)$ be an n -tuple of variables over \mathbb{F}_q , and let $\mathcal{F} = t^n + U_1 t^{n-1} + \dots + U_n \in \mathbb{F}_q[\mathbf{U}, t]$. For each $i = 1, \dots, r$, let $\mathcal{F}_i = \mathcal{F} + a_i$. We let $E = \tilde{\mathbb{F}}_q(\mathbf{U})$, for each $i = 1, \dots, r$, we let F_i be the splitting field of \mathcal{F}_i over E , and we denote by F the compositum of F_1, \dots, F_r . Thus F is the splitting field of $\prod_{i=1}^r \mathcal{F}_i$ over E . Since each of the \mathcal{F}_i has variable coefficients, we have an isomorphism

$$(1) \quad \text{Gal}(F_i/E) \cong S_n$$

that is induced from the action of $\text{Gal}(F_i/E)$ on the roots of \mathcal{F}_i in F_i (here S_n is the symmetric group of degree n). Thus the restriction maps $r_i: \text{Gal}(F/E) \rightarrow \text{Gal}(F_i/E)$, $i = 1, \dots, r$, induce an embedding $\varphi: \text{Gal}(F/E) \rightarrow \prod_{i=1}^r \text{Gal}(F_i/E) \cong S_n^r$.



Proposition 3.1. *Under the notation above, $\varphi: \text{Gal}(F/E) \rightarrow S_n^r$ is an isomorphism.*

The proof the Proposition 3.1 appears after the following two lemmas.

For a separable polynomial $f \in E[t]$ we denote by $\delta_t(f)$ the square class of its discriminant $\text{disc}_t(f)$ in the \mathbb{F}_2 -vector space $E^\times/(E^\times)^2$. (Recall that the characteristic of E is not 2.)

Lemma 3.2 (Carmon-Rudnick). *The number N of $\mathbf{u} = (u_1, \dots, u_n - 1) \in \mathbb{F}_q^{n-2}$ for which $\text{disc}_t(\mathcal{F}(\mathbf{u}, U_n, t) + a_1), \dots, \text{disc}_t(\mathcal{F}(\mathbf{u}, U_n, t) + a_r)$ are square-free, coprime, and non-constant is $q^{n-1} + O_{n,r}(q^{n-2})$.*

Proof. In [4, Page 3], for some i , a set G_n is defined to be the tuples \mathbf{u} such that $\text{disc}_t(\mathcal{F}(\mathbf{u}, U_n, t) + a_i)$ is square-free of positive degree, and for every $j \neq i$, $\text{disc}_t(\mathcal{F}(\mathbf{u}, U_n, t) + a_j)$ and $\text{disc}_t(\mathcal{F}(\mathbf{u}, U_n, t) + a_i)$ are coprime. They show $\#G_n = q^{n-1} + O_{n,r}(q^{n-2})$.

Let us denote the above set by $G_{n,i}$ to keep track of the dependence of i . Then $N = \#(\bigcap_{i=1}^n G_{n,i}) = q^{n-1} + O_{n,r}(q^{n-2})$, as claimed. \square

Lemma 3.3. *The square classes $\delta_t(\mathcal{F}_1), \dots, \delta_t(\mathcal{F}_r)$ are linearly independent in $E^\times/(E^\times)^2$.*

Proof. For a specialization $\epsilon: (U_1, \dots, U_{n-1}) \mapsto (u_1, \dots, u_{n-1}) \in \mathbb{F}_{q^e}^{n-1}$ we let $f_\epsilon = t^n + u_1 t^{n-1} + \dots + u_{n-1} t + U_n \in \mathbb{F}_{q^e}[U_n, t]$ be the specialized polynomial. By Lemma 3.2 for $q^{e(n-1)} + O_{n,r}(q^{e(n-2)})$ such specializations ϵ the discriminants $\text{disc}_t(f_\epsilon + a_1), \dots, \text{disc}_t(f_\epsilon + a_r)$ are square free, coprime, and non-constant. In particular if e is sufficiently large there exists at least one such specialization in $\mathbb{F}_{q^e} \subseteq E$. Thus the square classes $\delta_t(f_\epsilon + a_i)$ of $\text{disc}_t(f_\epsilon + a_i)$ in the \mathbb{F}_2 -vector space $\mathbb{F}_q(U_n)^\times/(\mathbb{F}_q(U_n)^\times)^2$ are linearly independent. Since squares cannot be specialized to non-squares, we get that $\delta_t(\mathcal{F}_1), \dots, \delta_t(\mathcal{F}_r)$ are linearly independent. \square

Another ingredient in the proof of Proposition 3.1 is the following linearly disjointness criterion that is proved in [1, Lemma 3.4]:

Lemma 3.4. *If $\text{Gal}(\mathcal{F}_i, E) = \text{Gal}(F_i/E) = S_n$ for all $i = 1, \dots, r$ and if $\delta_t(\mathcal{F}_1), \dots, \delta_t(\mathcal{F}_r)$ are linearly independent in $E^\times/(E^\times)^2$, then F_1, \dots, F_r are linearly disjoint over E .*

Proof of Proposition 3.1. Lemma 3.3 and (1) imply that the conditions of Lemma 3.4 are met. Therefore F_1, \dots, F_r are linearly disjoint, hence φ is an isomorphism ([6, VI.1.15, §1]). \square

4. PROOF OF THEOREM 1.1

Let n and r be fixed positive integers and let q be an odd prime power. Let $a_1, \dots, a_r \in \mathbb{F}_q[t]$ be distinct polynomials, each of degree $< n$. Let $\mathbf{U} = (U_1, \dots, U_n)$ be an n -tuple of variables over \mathbb{F}_q , and let $\mathcal{F} = t^n + U_1 t^{n-1} + \dots + U_n \in \mathbb{F}_q[\mathbf{U}, t]$. For each $i = 1, \dots, r$, let $\mathcal{F}_i = \mathcal{F} + a_i$. Proposition 3.1 shows that the assumptions of Proposition 2.1 are met.

Thus the number of $\mathbf{u} \in \mathbb{F}_q$ for which $\mathcal{F}(\mathbf{u}, t) + a_i = \mathcal{F}_i(\mathbf{u}, t)$ are simultaneously irreducible in $\mathbb{F}_q[t]$ is $\frac{q^n}{n^r} + O_{n,r}(q^{n-\frac{1}{2}})$. This finishes the proof since this number equals $\pi(q, n; \mathbf{a})$. \square

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